divisor but not right zero divisor.

1.1.3.5. Ring Without Zero Divisors

A ring (R, +,.) is called a ring without zero divisors if $a \cdot b = 0$ for a, b $\in$ R then either $a = 0$ or $b = 0$.

Or

A ring (R· +,·) is without zero divisor if the product of two elements a, b $\in$ R is such that if $a \cdot b = 0$, then it implies that $a = 0$, or $b = 0$ or both a and b are zero.

For example, Since the product of two non-zero integers is never zero so the ring of integers is a ring without zero divisors.

Theorem 7: A ring R is without zero divisors if and only if the cancellation laws of multiplication hold R.

Proof: Let (R, +,·) be a ring without zero divisors. Now to prove cancellation laws hold in R let a, b, c $\in$ R such that $ab = ac$ and $a \neq 0$

$ab - ac \Rightarrow a(b - c)$

Since, $a \neq 0$ and R has no zero divisors then

Thus, $b - c = 0$ or $b = c$

Hence if $a \neq 0$ then $ab = ac \Rightarrow b = c$

Since, $a \neq 0$ and R has no zero divisors then we have

$\therefore b - c = 0$ or $b = c$

Hence if $a \neq 0$ then $ab = ac \Rightarrow b = c$

Therefore left cancellation law is proved.

Suppose cancellation laws hold in R hence to prove that R does not have zero divisors. On the contrary assume R has zero divisors.

i.e., $ab = 0$ and $a \neq 0, b \neq 0$

From left cancellation law, we have

$\Rightarrow b = 0$ (left cancellation law)

Since $b \neq 0$, this is contradiction. Also if $ab = 0$, $b \neq 0 \Rightarrow a = 0$ again a contradiction.

Hence, cancellation law holds in R. It is free from zero divisors.

1.1.3.6. Commutative Ring

Commutative ring are those ring which satisfies commutative law. Conversely, if $a \cdot b = b \cdot a \ \forall \ a, b \in R$ then a ring (R, +, .) is known as a commutative ring.

Example 3: Show that the set I (or z) of all integers with ordinary addition and multiplication as composition forms a ring.

Solution:

1) **R_1: At first prove (I, +) is an Abelian group.**
   i) The sum of two integers is also an integer. So (I, +) is closed under addition.
   ii) For all the integers the associative law for addition holds.
   iii) Identity element is zero integer for,

      $0 + a = a + 0 = a \quad \forall \ a \in I$

   iv) Inverse of integer a is –a, for

      $a + (-a) = 0 \quad \forall \ a \in I$

   v) All integer commute under addition. Hence (I, +) is an abelian group.

2) **R_2: (Now prove (I,·) is a semi-group)**
   i) The multiplication of two integers is again an integer therefore I is closed under multiplication.
   ii) Integers follow associative law for multiplication. Therefore (I, ·) is a semi group.